Instructions: Answer 7 of the following 9 questions. All questions are of equal weight. Indicate clearly on the first page which questions you want marked.

1. Answer both parts.

(a) What does it mean to say that a utility function, \( u(\cdot) \), represents a preference relation on some choice set \( X \)? Prove that if \( u(\cdot) \) represents preference relation \( \succeq \), this preference relation must be complete and transitive.

(b) Now suppose \( X = \mathbb{R}^2_+ \) and \( (x_1^1, x_2^1) \succ (x_1^2, x_2^2) \) when \( x_1^1 > x_1^2 \), or \( x_1^1 = x_1^2 \) and \( x_2^1 > x_2^2 \). Is this preference relation complete, transitive and continuous? Defend your answers.

ANSWER

(a) A utility function \( u : X \rightarrow \mathbb{R} \) represents a preference relation \( \succeq \) on the choice set \( X \) if

\[
\forall x, y \in X, \ x \succeq y \iff u(x) \geq u(y).
\]

Completeness: We need to show that \( \forall x, y \in X \), either \( x \succeq y \) or \( y \succeq x \) or both. Now note that \( u(x) \) and \( u(y) \) are real numbers so either \( u(x) \geq u(y) \) in which case \( x \succeq y \) or \( u(y) \geq u(x) \) in which case \( y \succeq x \); if \( u(x) = u(y) \) we know both are true.

Transitivity: Suppose \( x, y, z \in X \), and \( x \succeq y \) and \( y \succeq z \). We need to show \( x \succeq z \). Since \( u \) represents \( \succeq \), \( u(x) \geq u(y) \) and \( u(y) \geq u(z) \). Since these are real numbers \( u(x) \geq u(z) \Rightarrow x \succeq z \).

(b) This is an example of lexicographic preferences where good 1 is the dominant good. Lexicographic preferences are complete and transitive but not continuous.

Completeness: Consider any two distinct points in \( \mathcal{R}^2_+ \) — \((x_1^1, x_2^1)\) (point \(a\)) and \((x_1^2, x_2^2)\) (point \(b\)). If \( x_1^1 > x_1^2 \), \(a \succ b\). If \( x_1^2 > x_1^1 \), \( b \succ a \). If \( x_1^1 = x_1^2 \), since \(a\) and \(b\) are distinct, it must be that either \( x_2^1 > x_2^2 \) in which case \( a \succ b \) or \( x_2^1 > x_2^2 \) in which case \( b \succ a \).

Transitivity: Let \(a, b, c \in \mathcal{R}^2_+ \) where \(a = (x_1^1, x_2^1)\), \(b = (x_1^2, x_2^2)\) and \(c = (x_1^3, x_2^3)\). Suppose \(a \succ b\) and \(b \succ c\). We want to prove \(a \succ c\). Now notice that \(a \succ b\) and \(b \succ c\) \(\Rightarrow x_1^1 \geq x_1^2\).
There are two cases (i) and (ii). In case (i), \( x_1^1 = x_3^1 \) and then we know that \( x_1^1 = x_1^2 = x_3^3 \) in which case it must be that \( x_1^1 > x_2^2 > x_3^3 \), and thus \( x_1^1 > x_2^2 \) which means \( a > c \). In case (ii) \( x_1^1 > x_3^1 \) and then immediately we know \( a > c \).

Continuity: Suppose
\[
\{x_n\} \subset \mathbb{R}_+^2 \text{ and } \lim_{n \to \infty} x_n = x
\]
and
\[
\{y_n\} \subset \mathbb{R}_+^2 \text{ and } \lim_{n \to \infty} y_n = y
\]
and \( x_n \succ y_n, \forall n \). Then if \( \succ \) were continuous we would be able to deduce that \( x \succ y \). Here is one counterexample. Let \( x_n = (1/n, 0) \) and \( y_n = (0, 1) \). Then \( x = (0, 0), y = (0, 1), x_n \succ y_n \forall n \) but \( y \succ x \). So lexicographic preferences are complete and transitive but they violate continuity.

2. This question applies what we have done in class this term to think about a carbon tax. Suppose a typical Canadian household currently buys 200 litres of gasoline per month at $1 per litre and spends $1800 per month buying goods (and services) other than gasoline. Assume the household’s choices are consistent with maximizing a Cobb-Douglas utility function in the two goods — “gasoline” and “dollars spent on goods other than gasoline”. Now suppose the government is considering a carbon tax that will raise the price of gasoline from $1 to $2 per litre. According to the EV measure of welfare change how much would the government have to pay this household to exactly compensate the household for having to pay the carbon tax? What would the answer be with CV?

ANSWER

Denote the price of gas by \( p_g \) and gas consumed by \( G \). The price of a dollar spent on other goods is 1 and let \( m \) be the number of dollars spent on goods other than gasoline; denote income/wealth by \( w \). With CD preferences we know the indirect utility function and the expenditure functions for
\[
u (g, m) = g^a m^{1-a}, \quad 0 < a < 1
\]
are
\[
V (p_g, 1, w) = a^a (1-a)^{1-a} p_g^{-a} (1-a)^{1-a} w \text{ and}
\]
\[
e (p_g, 1, u_0) = a^{-a} (1-a)^{-(1-a)} p_g a^{1-a} u_0
\]

The level of utility with \( p_g = 2 \) is clearly lower than the initial level of utility where \( p_g = 1 \). EV will be a negative number in this instance because utility goes down with the price increase. Since it’s easier to work with positive numbers think of EV as a positive
number. Then EV uses initial prices to measure the number of dollars it would take to move from the lower to the higher level of utility. Thus

$$EV = e(1, 1, V(1, 1, 2000)) - e(1, 1, V(2, 1, 2000))$$
$$= 2000 - 2^{-a}2000$$
$$= 2000(1 - 2^{-a})$$

CV (again, thought of as a positive number) is the same except that it uses the new price of gas.

$$CV = e(2, 1, V(1, 1, 2000)) - e(2, 1, V(2, 1, 2000))$$
$$= 2^a2000 - 2^a2^{-a}2000$$
$$= 2000(2^a - 1)$$

Determining $a$ is easy. We know

$$G = aw/p_g,$$

so with the initial values of $G = 200$, $w = 2000$, and $p_g = 1$ $a$ must be $1/10$.

3. Prove the following statements.

(a) Assume a two-good model. Prove the expenditure function, $e(p_1, p_2, u_0)$, must be concave in prices;

(b) In the choice-based approach, with two goods, the Slutsky matrix, $S$, must be symmetric.

**ANSWER**

(a) Consider two sets of prices $(p_1^1, p_2^1)$ and $(p_1^2, p_2^2)$ and the convex combination of them $(p_1^t, p_2^t) = t(p_1^1, p_2^1) + (1 - t)(p_1^2, p_2^2)$ for $0 \leq t \leq 1$. Then $e(p_1, p_2, u_0)$ is concave in prices $(p_1, p_2)$ if

$$e(p_1^t, p_2^t, u_0) \geq te(p_1^1, p_2^1, u_0) + (1 - t)e(p_1^2, p_2^2, u_0).$$

Let $(h_1^t, h_2^t)$ be the expenditure-minimizing bundle for prices $(p_1^t, p_2^t)$ and utility level $u_0$. Then

$$e(p_1^t, p_2^t, u_0) = p_1^t h_1^t + p_2^t h_2^t$$
$$= (tp_1^1 + (1 - t)p_1^2)h_1^t + (tp_2^1 + (1 - t)p_2^2)h_2^t$$
$$= t(p_1^1 h_1^t + p_1^2 h_2^t) + (1 - t)(p_2^1 h_1^t + p_2^2 h_2^t)$$
$$\geq te(p_1^1, p_2^1, u_0) + (1 - t)e(p_1^2, p_2^2, u_0).$$
The last line follows because the expenditure function minimizes the cost of buying utility.

(b) The choice-based approach assumes the demand system is homogeneous of degree zero and satisfies Walras’s Law. With two goods this means that the Slutsky substitution matrix, \( S \), satisfies these identities for all positive prices:

\[
\begin{bmatrix}
  p_1 & p_2 \\
  S_{11} & S_{12} \\
  S_{21} & S_{22}
\end{bmatrix} \begin{bmatrix}
  p_1 \\
  p_2
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  S_{11} & S_{12} \\
  S_{21} & S_{22}
\end{bmatrix} \begin{bmatrix}
  p_1 \\
  p_2
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

Thus from the first identity we know

\[ p_1 S_{11} + p_2 S_{21} = 0 \]

and from the second identity

\[ S_{11} p_1 + S_{12} p_2 = 0 \]

Putting these together we know that for any positive value of \( p_2 \)

\[ p_2 S_{21} = S_{12} p_2 \text{ or } S_{21} = S_{12} \]

4. I have talked with you about the importance of microeconometrics and we have spent considerable time on tax policy. This question combines these themes. Microdata on male labour supply show that men work pretty much the same hours per week whatever the wage rate, and whatever the marginal tax rate. These facts are consistent with two very different models. In model A, using standard notation and with proportional commodity tax rates in effect, the typical person’s budget constraint is

\[
(1 + t_1) p_1 x_1 + (1 + t_2) p_2 x_2 + wl = wT, \quad l \leq T
\]

and preferences are represented by the following utility function

\[ U (l, x_1, x_2) = l^a x_1^b x_2^{1-a-b}, \quad a > 0, b > 0, a + b < 1. \]

In model B, the firms people work for fix the hours worked per week for each employee, and given the choice of working at these fixed hours, or no job, people choose to work the hours specified by the employer, that is, time at work, \( T - l \), is fixed in model B. Suppose you have been hired by the government to advise them on commodity taxation — in particular, for economic efficiency, should \( t_1 > t_2 \) or \( t_2 > t_1 \) or \( t_1 = t_2 \)? Would your advice depend on whether model A or model B is the better characterization of reality? Defend your answer.
ANSWER

Advice based on model A:

Here we are in a second-best world where taxation is distortionary. Is it efficient for the government to set $t_1 = t_2$? We know that

$$t_1 = t_2 \text{ is efficient if and only if } \frac{\partial d(l, x_1, x_2, u_0)}{\partial x_1} \text{ is independent of } l,$$

where $d(l, x_1, x_2, u_0)$ is the distance function that corresponds to $U(l, x_1, x_2) = l^a x_1^b x_2^{1-a-b}$. To find the distance function apply the definition of the distance function and obtain

$$U\left(\frac{l x_1}{d}, \frac{x_2}{d}\right) = u_0 \text{ or }\left(\frac{l}{d}\right)^a \left(\frac{x_1}{d}\right)^b \left(\frac{x_2}{d}\right)^{1-a-b} = u_0 \text{ or } d(l, x_1, x_2, u_0) = u_0^{-1} l^a x_1^b x_2^{1-a-b}.$$

Then

$$\frac{\partial d(l, x_1, x_2, u_0)}{\partial x_1} = \frac{bu_0^{-1} l^a x_1^b x_2^{1-a-b}}{(1 - a - b) u_0^{-1} l^a x_1^b x_2^{1-a-b}} = \frac{bx_2}{(1 - a - b) x_1},$$

which is independent of $l$. Thus setting $t_1 = t_2$ is (second-best) efficient in model A.

Advice based on model B:

In model B, leisure is fixed so earnings are an endowment and then we know setting $t_1 = t_2$ is first-best efficient.

To summarize, both models support equal proportional taxation of consumption goods. The only difference between the models is that taxation will cause a deadweight loss in model A, and, as usual, the higher the consumption tax rate, the higher the ratio of DWL to tax revenue.

5. You are given the following information about a consumer’s purchases. Goods 1 and 2 are the only goods consumed.

<table>
<thead>
<tr>
<th></th>
<th>Year 1</th>
<th></th>
<th>Year 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Quantity</td>
<td>Price</td>
<td>Quantity</td>
<td>Price</td>
</tr>
<tr>
<td>Good 1</td>
<td>100</td>
<td>100</td>
<td>120</td>
<td>$p_1^2$</td>
</tr>
<tr>
<td>Good 2</td>
<td>100</td>
<td>100</td>
<td>$x_2^2$</td>
<td>80</td>
</tr>
</tbody>
</table>
For what values of $p_1^2$ will there exist values of $x_2^2$ that contradict the weak axiom of revealed preference? Justify your answer carefully.

**ANSWER**

The year 1 bundle is revealed preferred to the year 2 bundle if

$$(100)(100) + (100)(100) \geq (100)(120) + (100)x_2^2.$$  

The year 2 bundle is revealed preferred to the year 1 bundle if

$$p_1^2(120) + (80)x_2^2 \geq p_1^2(100) + (80)(100).$$

These two inequalities can be written as

$$80 \geq x_2^2$$

$$x_2^2 \geq 100 - \frac{1}{4}p_1^2$$

For WARP to be contradicted these two intervals must overlap so

$$80 \geq 100 - \frac{1}{4}p_1^2 \text{ or } p_1^2 \geq 80$$

Given that the person buys more of good 1 in year 2, if the relative price of good 1 goes up in year 2 then it’s possible that the data could contradict WARP.

6. Consider a price-taking consumer in a two-good world where the consumer is endowed with $w$ dollars of wealth. Write the indirect utility function as the maximized value of the appropriate Lagrangian expression and show that

$$\frac{\partial V}{\partial p_1}(p_1, p_2, w) \leq 0.$$  

Now consider the same consumer in a world where she is endowed with quantities of the two goods $(e_1, e_2)$ and no money. Fix $p_2, e_1$ and $e_2$ and describe as precisely as you can the relationship between the modified indirect utility function $V^*(p_1, p_2, e_1, e_2)$ and $p_1$.

**ANSWER**

We know

$$V(p_1, p_2, w) = \max_{x_1, x_2, \lambda} u(x_1, x_2) + \lambda(w - p_1x_1 - p_2x_2)$$

Applying the envelope theorem
\[
\frac{\partial V(p_1, p_2, w)}{\partial p_1} = -\lambda (p_1, p_2, w) x_1 (p_1, p_2, w).
\]

Since \(\lambda (p_1, p_2, w) \geq 0\) and \(x_1 (p_1, p_2, w) \geq 0\)

\[
\frac{\partial V(p_1, p_2, w)}{\partial p_1} \leq 0.
\]

Now

\[
V^* (p_1, p_2, e_1, e_2) = \max_{x_1, x_2} u(x_1, x_2) + \lambda (p_1 e_1 + p_2 e_2 - p_1 x_1 - p_2 x_2)
\]

Applying the envelope theorem

\[
\frac{\partial V^* (p_1, p_2, e_1, e_2)}{\partial p_1} = -\lambda (p_1, p_2, e_1, e_2) [e_1 - x_1 (p_1, p_2, e_1, e_2)].
\]

Since \(\lambda (p_1, p_2, e_1, e_2) \geq 0\),

\[
\text{Sign} \frac{\partial V^* (p_1, p_2, e_1, e_2)}{\partial p_1} = \text{Sign} x_1 (p_1, p_2, e_1, e_2) - e_1.
\]

Given the fixed \(p_2, e_1\) and \(e_2\) define \(p_1^*\) by

\[
x_1 (p_1^*, p_2, e_1, e_2) = e_1.
\]

Then a graph of \(V^*\) against \(p_1\) will have a U-shape with the bottom of the U at \(p_1^*\).

7. Consider the expected-utility maximizing individual discussed in class who must decide whether and for how much to insure his car. Assume the probability that he will not have an accident is \(\pi\). Denote his initial wealth by \(w_0\) and denote his loss in the event of an accident by \(L\). Suppose that insurance is available at a price \(p\) for \$1 worth of insurance coverage.

(a) Prove that if he is offered fair insurance he will choose to fully insure, that is, his wealth will be the same whether he has an accident or not.

(b) At the point where \(p\) is set to deliver fair insurance, does the amount of car insurance purchased increase or decrease with \(w_0\)? Justify your answer carefully.

**ANSWER**

(a) The individual’s expected utility is

\[
f (x, p, \pi, w_0, L) \equiv \pi u (w_0 - px) + (1 - \pi) u (w_0 - px - L + x)
\]

If the insurance is priced “fairly” the firm’s expected profit on each dollar of insurance is zero, so

\[
\pi p + (1 - \pi) (p - 1) = 0 \quad \text{or} \quad p = 1 - \pi.
\]
To find the optimal value of $x$ given the parameters of the problem set

$$\frac{\partial f(x, p, \pi, w_0, L)}{\partial x} = 0$$

and solve for $x$ at $p = 1 - \pi$. Thus

$$(-p) \pi u'(w_0 - px) + (1 - \pi) (1 - p) u'(w_0 - px - L + x) = 0$$

or

$$u'(w_0 - px) = u'(w_0 - px - L + x)$$

and thus

$$w_0 - px = w_0 - px - L + x$$

or

$$x = L.$$

Thus his wealth is $w_0 - (1 - \pi) L$ whether he has an accident or not.

(b) We have just proved that when $p = 1 - \pi$, the demand for insurance, $x(p, \pi, w_0, L)$, equals $L$, so

$$x(1 - \pi, \pi, w_0, L) = L,$$

which holds for all sensible values of $w_0$ at this point. Therefore

$$\frac{dx(1 - \pi, \pi, w_0, L)}{dw_0} = 0$$

8. In the two-type car insurance model discussed in class, when the insurance companies cannot distinguish the good from the bad drivers, prove that a pooling equilibrium cannot exist and explain why a separating equilibrium may not exist. If you use diagrams in your answer make sure they are carefully drawn and clearly labeled.

**ANSWER**

Let the candidate for a pooling equilibrium be at point A on the market line (the m-line). Since the good driver indifference curve through A must be steeper than the bad driver indifference curve through A there must be points, say like B, below the bad driver indifference curve and above the good driver indifference curve, southeast of A and below the good-driver zero-profit line. If contract B were offered it would attract only the good drivers and it would therefore make positive profits because it would attract only the good drivers (it is below the g-line). The firms holding the contract at A will now lose money because their pool of customers has disproportionately more bad drivers. Thus A cannot be sustained.

The best chance of finding a separating equilibrium puts all the bad drivers at their preferred point on the b-line, and the good drivers at the point where the bad driver indifference curve cuts the g-line; call this point X. If the m-line cuts the good driver indifference curve through X there cannot be a separating equilibrium. The reason is that there will be pooling contracts below the m-line and about the good driver indifference curve that will
attract both good and bad drivers and these contracts will also be profitable. These pooling contracts knock out the best chance for a separating equilibrium. Since there cannot be a pooling equilibrium, there is no trade — all drivers, good or bad, end up at their endowment point \((w_0, w_0 - L)\).

9. (a) The AP mill in Dryden Ontario produces wood pulp with labour, \(N\), capital, \(K\), and wood, \(W\). Assume AP acts to maximize its profits and is a price-taker in all input and output markets. Denote the input prices by \(P_N\), \(P_K\), \(P_W\) and the output price by \(P\). Using data on profits, \(\pi\) (which are always positive) suppose one of your colleagues has estimated AP’s profit function using the following functional form:

\[
\ln \pi = \alpha_0 + \alpha_1 \ln P_N + \alpha_2 \ln P_K + \alpha_3 \ln P_W + \alpha_4 \ln P.
\]

Since AP is a major employer in Dryden and there is a lot of unemployment in the area the Ontario Ministry of Finance has decided to subsidize at a proportional rate, \(\lambda\), the price of labour to the firm. Assume AP’s data fit the above profit function perfectly and no price changes as a consequence of the wage subsidy. Find a formula for \(\lambda\) in terms of the parameters of the profit function if the government’s objective is to increase AP’s use of labour by 50 percent.

(b) Suppose a price-taking firm can produce output \(q\) with inputs \(z_1\) and \(z_2\). When the output price is \(p\) and input prices are \(w_1\) and \(w_2\), the profit-maximizing levels of output and inputs are \(q^*, z_1^*\) and \(z_2^*\). Suppose in the “short run” \(z_2\) cannot be varied but \(z_1\) can be, but in the “long run” both inputs are variable. Let the price of output rise to \(p' > p\), and let input prices be constant. Prove that the increase in output in the long run is at least as large as the increase in output in the short run.

**Answer**

(a) Since

\[
\pi(P_N, P_K, P_W, P) = \max_{N, K, W} Pf(N, K, W) - (P_N N + P_K K + P_W W),
\]

where \(f\) is the production function, we can use the envelope theorem to obtain the input demand for \(N\) as a function of the input prices and the output price.

\[
N(P_N, P_K, P_W, P) = -\frac{\partial \pi(P_N, P_K, P_W, P)}{\partial P_N}
\]

Taking the derivative of the profit function with respect to \(P_N\) we obtain

\[
\frac{1}{\pi} \frac{\partial \pi}{\partial P_N} = \frac{\alpha_1}{P_N} \text{ or } N(P_N, P_K, P_W, P) = -\frac{\alpha_1 \pi}{P_N}.
\]

Since
\[ \pi = e^{\alpha_0} P_N^{\alpha_1} P_K^{\alpha_2} P_W^{\alpha_3} P^4 \]

we can see that

\[ N(P_N, P_K, P_W, P) = -\alpha_1 e^{\alpha_0} P_N^{\alpha_1-1} P_K^{\alpha_2} P_W^{\alpha_3} P^4. \]

So to increase the firm’s use of \( N \) by 50 percent we need

\[ \frac{3}{2} = (1 + \lambda)^{\alpha_1-1}, \]

which can be solved for \( \lambda \).

(b) Write short-run total cost as \( C_S(q) \) and long-run total cost as \( C_L(q) \). In \( C_S(q) \) \( z_2 \) is fixed at \( z_2^* \) and so \( C_S(q) \geq C_L(q) \) and we know the two are equal at \( q^* \). So \( q^* \) is a minimizer of \( g(q) \equiv C_S(q) - C_L(q) \). Thus

\[ g'(q^*) = C_S'(q^*) - C_L'(q^*) = 0 \]
\[ g''(q^*) = C_S''(q^*) - C_L''(q^*) > 0. \]

So at \( q^* \) the short-run and long-run marginal costs are equal to each other and the short-run MC is steeper than the long-run MC. Given \( p' > p \), then, the \( q \) at \( p' = C_L'(q) \) will exceed the \( q \) at \( p' = C_S'(q) \).