1 The model

Assume two types — A, B — two goods — X, Y. Denote A’s utility by \( u^A(x^A, y^A) \) and B’s utility by \( u^B(x^B, y^B) \). Write the production possibility boundary as \( F(X, Y) = 0 \). Taking a total differential

\[
F_X dX + F_Y dY = 0 \quad \text{or} \quad \frac{dY}{dX} = \frac{F_X}{F_Y} = \frac{\text{MC of } X}{\text{MC of } Y}.
\]

2 Both X and Y private goods

To find equations that describe Pareto efficient allocations we can maximize the utility of B subject to some fixed level of utility for A and the production possibility boundary. Note that \((x^A, y^A)\) can be written as \((X - x^B, Y - y^B)\)

\[
\begin{align*}
\text{Max} & \quad u^B(x^B, y^B) + \lambda(\underbrace{u^A(X - x^B, Y - y^B) - u_0^A}_{\text{subject to}}) + \mu F(X, Y) \\
\text{w.r.t.} & \quad x^B, y^B, X, Y, \lambda, \mu
\end{align*}
\]

The first-order conditions for \((x^B, y^B, X, Y)\) are

\[
\begin{align*}
\frac{\partial u^B}{\partial x^B} - \lambda \frac{\partial u^A}{\partial x^A} &= 0 \quad (1) \\
\frac{\partial u^B}{\partial y^B} - \lambda \frac{\partial u^A}{\partial y^A} &= 0 \quad (2) \\
\lambda \frac{\partial u^A}{\partial x^A} + \mu F_X &= 0 \quad (3) \\
\lambda \frac{\partial u^A}{\partial y^A} + \mu F_Y &= 0 \quad (4)
\end{align*}
\]

Recall that the \(\text{MRS}_{XY} = \frac{\text{MU}_X}{\text{MU}_Y}\). Take the \(\lambda\) and \(\mu\) terms to the other side of the equations and divide (1) by (2), and (3) by (4). Then we can see that in a Pareto efficient allocation, with two private goods

\[
\text{MRS}_{XY}^A = \text{MRS}_{XY}^B = \frac{\text{MC of } X}{\text{MC of } Y}.
\]
3 One public good, one private good

In the previous section $X$ was a private good. In this section I replace it with $G$ a pure public good — if $G$ units of a public good are produced, $G$ units of the good can be consumed by each person — consumption of the good is said to be “non-rival”; one person’s consumption of the good does not reduce the amount available for anyone else. I continue to assume $Y$ is a private good. Denote the new production possibility boundary by $H(G, Y) = 0$. To find Pareto-efficient allocations we can solve

$$\begin{align*}
\text{Max} & \quad B_G(G, y^B, Y) + \lambda (u^A(G, Y - y^B) - u^A_0) + \mu H(G, Y)
\end{align*}$$

The first-order conditions for $(G, y^B, Y)$ are

$$\begin{align*}
\frac{\partial u^B}{\partial G} + \lambda \frac{\partial u^A}{\partial G} + \mu \frac{\partial H}{\partial G} &= 0 \quad (5) \\
\frac{\partial u^B}{\partial y^B} - \lambda \frac{\partial u^A}{\partial y^A} &= 0 \quad (6) \\
\lambda \frac{\partial u^A}{\partial y^A} + \mu \frac{\partial H}{\partial Y} &= 0. \quad (7)
\end{align*}$$

Rewrite (5), (6) and (7) as

$$\begin{align*}
\frac{\partial u^B}{\partial G} + \lambda \frac{\partial u^A}{\partial G} &= -\mu \frac{\partial H}{\partial G} \quad (8) \\
\frac{\partial u^B}{\partial y^B} &= \lambda \frac{\partial u^A}{\partial y^A} \quad (9)
\end{align*}$$

Now divide each term in (8) by the corresponding term in (9) to obtain:

$$\text{MRS}^B_{GY} + \text{MRS}^A_{GY} = \frac{\text{MC of } G}{\text{MC of } Y}.$$ 