1 The model

Assume two types — \( A, B \) — two goods — 1, 2. Denote \( A \)'s utility by \( u^A(x^A_1, x^A_2) \) and \( B \)'s utility by \( u^B(x^B_1, x^B_2) \). Consider a pure exchange economy with total endowments of \((e_1, e_2)\).

2 Pareto efficiency with no externalities

To find equations that describe Pareto efficient allocations we can maximize the utility of \( B \) subject to some fixed level of utility for \( A \). Note that \((x^A_1, x^A_2)\) can be written as

\[
(e_1 - x^B_1, e_2 - x^B_2)
\]

The first-order conditions for \((x^B_1, x^B_2)\) are

\[
\begin{align*}
u^B_1(x^B_1, x^B_2) - \lambda u^A_1(e_1 - x^B_1, e_2 - x^B_2) &= 0 & (1) \\
u^B_2(x^B_1, x^B_2) - \lambda u^A_2(e_1 - x^B_1, e_2 - x^B_2) &= 0 & (2)
\end{align*}
\]

Recall that the \( MRS_{12} = MU_1/MU_2 \). Take the \( \lambda \) term to the other side of the equations and divide (1) by (2). Then we can see that in a Pareto efficient allocation, with two private goods

\[
MRS^A_{12} = MRS^B_{12}.
\]

3 Pareto efficiency with an externality

Suppose \( B \)'s utility is affected by \( A \)'s consumption of good 1. Write \( B \)'s utility as

\[
u^B(x^B_1, x^B_2, x^A_1) = u^B(x^B_1, x^B_2, e_1 - x^B_1).
\]

To find Pareto-efficient allocations we can solve
Opt
\[ x_1^B, x_2^B \ u^B (x_1^B, x_2^B, e_1 - x_1^B) + \lambda (u^A (e_1 - x_1^B, e_2 - x_2^B) - u_0^A) \]

Now the first-order conditions for \((x_1^B, x_2^B)\) are

\[
\begin{align*}
    u_1^B (x_1^B, x_2^B, e_1 - x_1^B) - u_3^B (x_1^B, x_2^B, e_1 - x_1^B) - \lambda u_1^A (e_1 - x_1^B, e_2 - x_2^B) &= 0 \\
    u_2^B (x_1^B, x_2^B, e_1 - x_1^B) - \lambda u_2^A (e_1 - x_1^B, e_2 - x_2^B) &= 0
\end{align*}
\]

Dropping the arguments of the functions and proceeding as above obtain

\[
\frac{u_1^B - u_3^B}{u_2^B} = \frac{u_1^A}{u_2^A}
\]

Now it is no longer efficient to operate at allocations where the MRS for the two people is the same. If \(B\) benefits from \(A\)'s consumption of good 1, that is, \(u_3^B > 0\), we say there is a positive externality and it is efficient that the MRS for \(A\) be less than the MRS for \(B\). \(B\) might bribe \(A\) to consume more good 1. If \(B\) dislikes \(A\)'s consumption of good 1 then \(u_3^B < 0\) and we say there is a negative externality. Here it is efficient that the MRS for \(A\) exceed the MRS for \(B\). \(B\) might bribe \(A\) to consume less good 1.