Economics 393
The Elasticity of Substitution

Suppose the production function \( Y = F(K, N) \) is homogeneous of degree one. Then it can be written in intensive form as \( y = f(k) \) where \( y \equiv Y/N \) and \( k \equiv K/N \). Assume perfectly competitive markets for labour and capital; let \( r \) be the marginal product of capital and \( w \) the marginal product of labour. Then \( r = f'(k) \) and \( w = f(k) - kf''(k) \). The elasticity of substitution in this context, say \( \sigma \), is defined by the proportional change in the capital-labour ratio divided by the proportional change in \( w/r \).

\[
\sigma \equiv \frac{d \ln k}{d \ln (w/r)} = \frac{1}{k} \frac{w}{r} \frac{1}{d(w/r)} = \frac{w}{rk} \frac{dk}{d(w/r)}.
\]

But

\[
\frac{dr}{dk} = f''(k)
\]
\[
\frac{dw}{dk} = f'(k) - kf''(k) = f'(k) - f'(k) - kf''(k) = -kf''(k)
\]
\[
\frac{d(w/r)}{dk} = -kf''(k) \frac{f'(k)}{f(k)^2} - \frac{(f(k) - kf'(k)) f''(k)}{f'(k)^2} = -\frac{f(k) f''(k)}{f'(k)^2}.
\]

Then

\[
\sigma = \frac{w}{rk} \frac{dk}{d(w/r)}
\]
\[
= -\frac{f(k) - kf'(k)}{kf'(k)} \frac{f'(k)^2}{f(k) f''(k)}
\]
\[
= -\frac{f'(k)(f(k) - kf'(k))}{kf(k) f''(k)}
\]
\[
= -\frac{rw}{kf(k) f''(k)}.
\]
A Cobb-Douglas example

\[ Y = F(K, N) = K^\beta N^{1-\beta} \quad 0 < \beta < 1 \]

\[ \frac{Y}{N} = f(k) = k^\beta \]

\[ r = f'(k) = \beta k^{\beta-1} \]

\[ f''(k) = \beta(\beta - 1)k^{\beta-2} \]

\[ w = f(k) - kf'(k) = (1 - \beta)k^\beta \]

\[ \sigma = \frac{rw}{kf(k)f''(k)} = \frac{\beta k^{\beta-1}(1 - \beta)k^{\beta}}{kk^\beta(\beta - 1)k^{\beta-2}} = 1. \]