Instructions: You need paper (lined if possible) and a pen or a pencil to write this test. You may answer the questions in any order you like. You should start each question on a new page. You must write your answers; typed answers will not be accepted. When you are finished answering the questions, please order the pages so your answers to question 1 are first, and then your answers to question 2, etc. Then, in a single email message, send an image of each page to me at jburbidg@uwaterloo.ca. Please put Econ 393, your name and your id number in the subject line of your email. The deadline for submitting your answers is 6:00 pm Tuesday July 7th, Toronto time. The marks allocated to each question are shown in brackets.

1. Suppose the citizens of the City of Toronto care about only two things — aerobics lessons (A) and bread (B). Everyone has the same utility function

\[ U = AB. \]

The price of an aerobics lesson is $2 and the price of a loaf of bread is $1. Two million of Toronto’s citizens are poor and have an income of $50 each; one million are rich and have an income of $100 each. Fill out a table like the one below for each of the following cases.

(i) (2 marks) The private equilibrium with no government intervention.

(ii) (2 marks) The government decides to provide aerobics lessons publicly. Assume that majority vote determines the number of aerobics lessons, which must be the same for every citizen. Further assume that the government must balance its budget with a head tax.

(iii) (3 marks) Everything is as it was in (ii) except the government uses a proportional income tax.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Poor A B Utility</th>
<th>Rich A B Utility</th>
</tr>
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<tbody>
<tr>
<td>Private eq.</td>
<td>A B Utility</td>
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<td></td>
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<td>Income tax</td>
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</table>
ANSWER

(i) The utility function for this question is in the Cobb-Douglas family, \( u(A, B) = A^a B^b \) and we know

\[
A(p_A, p_B, \text{wealth}) = \frac{a}{a+b} \frac{\text{wealth}}{p_A} = \frac{1}{2} \frac{\text{wealth}}{2}
\]

\[
B(p_A, p_B, \text{wealth}) = \frac{b}{a+b} \frac{\text{wealth}}{p_B} = \frac{1}{2} \frac{\text{wealth}}{1}.
\]

The private-equilibrium row in the table below follows directly from these equations.

(ii) For the government in this question revenue must equal expenditure. With a head tax this means

\[
(3 \text{ million})(\text{head tax}) = (3 \text{ million})p_A A \quad \text{or, the head tax} = 2A.
\]

Since the poor have a majority their budget constraint is the one that matters.

\[
B = \text{Income} - \text{head tax} = 50 - 2A \quad \text{or} \quad 2A + B = 50.
\]

With this budget constraint the poor choose \( A = 25/2 \), with a head tax of 25, and \( B = 25 \). This means the rich have to live with \( A = 25/2 \) and \( B = 75 \). These results are summarized in the head-tax row of the table.

(iii) With a proportional income tax, \( t \), we have

\[
t[(1 \text{ million})(\text{income of rich}) + (2 \text{ million})(\text{income of poor})] = (3 \text{ million})2A,
\]

or

\[
t = \frac{6A}{200} = \frac{3A}{100}.
\]

So then the budget constraint of the poor is

\[
B = (1 - t) \text{ income} = 50 \left(1 - \frac{3A}{100}\right) = \frac{1}{2} (100 - 3A)
\]

or

\[
3A + 2B = 100.
\]

Given this budget constraint the poor vote for

\[
A = \frac{1}{2} \frac{100}{3} = \frac{50}{3} \quad \text{and} \quad t = \frac{3A}{100} = \frac{1}{2}.
\]

From here the results in the last row of the table follow directly.
2. Suppose the Government of Ontario has declared the roof of the old rink in New Liskeard to be unsafe and has condemned the building. The town has decided to hold a referendum on building a new rink. If the referendum passes everyone’s income will be taxed at a proportional rate $0 < t < 1$ to cover the costs of a new rink. If the rink is of size $G$ square feet, the total costs are $p_G G$. Everyone of the $N$ citizens of New Liskeard has a wealth of $w$. Apart from the rink there is a private good, $X$, which has a price of one dollar per unit of $X$. Everyone’s utility function is

$$U(X, G) = X + f(G), f'(G) > 0, f''(G) < 0.$$  

(i) (2 marks) What is the Pareto efficient level of $G$?

(ii) (3 marks) Will the referendum lead to the Pareto efficient level of $G$? Defend your answer carefully.

**ANSWER**

(i) The Pareto efficient level of $G$ is given by the Samuelson condition, that is, that the sum of the MRS of $G$ for $X$ should equal the marginal cost of $G$ divided by the marginal cost of $X$. Here this means that

$$\sum_{i=1}^{i=N} f'(G) = P_G \text{ or } Nf'(G) = P_G.$$  

(ii) If the town uses a proportional income tax to fund $G$ then $Ntw = P_G G$. So then we can deduce each citizen’s budget constraint.

$$X = (1 - t)w = w - tw = w - \frac{p_G G}{N},$$  

$$X + \frac{p_G}{N} G = w.$$  

For any resident, in effect the price of $G$ is $p_G/N$ and the price of $X$ is 1. To maximize utility each person will vote for that level of $G$ that makes her MRS$_{GX} = f'(G) = P_G/N$. This equation delivers the same level of $G$ as was obtained in (i). So, yes, on the assumptions made in this question (especially everyone has the same income), this proportional tax mechanism will deliver the Pareto efficient level of $G$. 

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</tr>
<tr>
<td>Income tax</td>
<td>50/3</td>
<td>25</td>
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</table>
3. A factory pumps waste into a lake. If $X$ is the amount of waste, the factory’s profits are

$$20X - X^2.$$ 

The lake may also be used for recreational swimming by 2 people, each of whom has the utility function

$$u(C, Y, X) = C + 8Y - Y^2 - XY,$$

where $C$ is dollars spent on consumption goods, $Y$ is hours per day spent swimming in the lake and, as above, $X$ is the amount of waste the factory pumps into the lake. Assume each person has 40 dollars to spend per day and swimming in the lake is free.

(i) (2 marks) Calculate factory profits ($\pi$), the utility of each person ($u$), and social utility $Z \equiv \pi + 2u$ in the private equilibrium.

(ii) (2 marks) What is the socially efficient level of $X$? Justify your answer.

**ANSWER**

(i) The externality runs from the firm to the consumers so we start with the firm. The firm chooses $X$ to maximize its profits. Solving $20 - 2X = 0$ we see that the firm chooses $X = 10$. Then, given each consumer’s utility function and the information to this point,

$$u(40, Y, 10) = 40 + 8Y - Y^2 - 10Y = 40 - 2Y - Y^2.$$ 

Making $Y$ positive would reduce each person’s utility, so they choose $Y = 0$. Then $\pi = 100$, $u = 40$ and $Z = 180$.

(ii) Since the utility function is quasilinear in $C$ and $C$ is dollars of consumption the social planner’s objective function can be obtained by adding total profits to total utility.

$$Z(X, Y) = 20X - X^2 + 2(40 + 8Y - Y^2 - XY)$$

$IF$ this function is strictly concave in $X$ and $Y$ for the space where $X, Y \geq 0$, one can find the socially-optimal levels of $X$ and $Y$ by setting its partial derivatives with respect to $X$ and $Y$ equal to zero and solving two equations in two unknowns. Calculate

$$

Z_1(X, Y) = 20 - 2X - 2Y \\
Z_2(X, Y) = 2(8 - 2Y - X) \\
Z_{11}(X, Y) = -2 \\
Z_{22}(X, Y) = -4 \\
Z_{12}(X, Y) = -2

$$

and then observe that the Hessian of $Z(X, Y)$ is negative definite which is a sufficient condition for a strictly concave function.

$$

\text{Hessian of } Z(X, Y) = \begin{pmatrix} -2 & -2 \\ -2 & -4 \end{pmatrix}

$$
The main diagonal is negative and the determinant is \((-2)(-4) - (-2)^2 = 4 > 0\) so \(Z(X, Y)\) is strictly concave. Setting the partial derivatives of \(Z\) with respect to \(X\) and \(Y\) equal to zero and solving two equations in two unknowns we obtain \(X = 12\) and \(Y = -2\). So even though social utility is strictly concave in \(X\) and \(Y\) it peaks where \(Y\) is negative — but negative hours of swimming is physically impossible. The social planner would set \(Y = 0\) and the socially optimal level of \(X\) is 10. In this setting, even though there is an externality, the private equilibrium is Pareto efficient.

4. For question 1 on assignment 8, calculate the competitive equilibrium values of the following derivatives:
   (i) (2 marks) \(dx/dp\);
   (ii) (2 marks) \(dx/dw_0\).

**ANSWER**

(i) From assignment 8 expected utility is

\[
f(x, p, \pi, w_0, L) \equiv \pi u(w_0 - px) + (1 - \pi) u(w_0 - px - L + x)
\]

And the derivative of expected utility with respect to \(x\) implicitly determines \(x\) as a function of all the parameters in the model.

\[
f_1(x, p, \pi, w_0, L) = (-p) \pi u'(w_0 - px) + (1 - \pi) (1 - p) u'(w_0 - px - L + x) = 0
\]

Taking a total differential of this equation for \(x, p\) and \(w_0\) we have

\[
f_{11}(x, p, \pi, w_0, L) dx + f_{12}(x, p, \pi, w_0, L) dp + f_{14}(x, p, \pi, w_0, L) dw_0 = 0.
\]

Drop the arguments of the functions to reduce clutter and calculate

\[
f_{11} = (-p)^2 \pi u''(w_0 - px) + (1 - \pi) (1 - p)^2 u''(w_0 - px - L + x)
\]
\[
f_{12} = -\pi u'(w_0 - px) + (p) \pi u''(w_0 - px) (-x)
\]
\[
- (1 - \pi) u'(w_0 - px - L + x) + (1 - \pi) (1 - p) u''(w_0 - px - L + x) (-x)
\]
\[
f_{14} = (-p)\pi u''(w_0 - px) + (1 - \pi)(1 - p)u''(w_0 - px - L + x)
\]

In competitive equilibrium \(p = 1 - \pi\) and \(x = L\) so we obtain

\[
f_{11} = \pi(1 - \pi)u''(w_0 - (1 - \pi) L)
\]
\[
f_{12} = -u'(w_0 - (1 - \pi) L)
\]
\[
f_{14} = 0
\]

Therefore

\[
\frac{dx}{dp} = -\frac{f_{12}}{f_{11}} = \frac{u'(w_0 - (1 - \pi) L)}{\pi(1 - \pi)u''(w_0 - (1 - \pi) L)}
\]
(ii) From above $dx/dw_0 = 0$.

5. (3 marks) Briefly explain the role the independence axiom plays in the expected utility theorem.

**ANSWER**

See pages 171 and 172 in MWG. While the independence axiom would make no sense in the standard consumer model in Econ 290, it seems appropriate in the uncertainty framework discussed in chapter 6. And it lies at the heart of the expected utility theorem which says that, among other things, expected utility is linear in the probabilities of the outcomes.